

Vertex cover problem studied by cavity method: Analytics and population dynamics

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Received 11 November 2002

Published online 1st April 2003 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2003

Abstract. We study the vertex cover problem on finite connectivity random graphs by zero-temperature cavity method. The minimum vertex cover corresponds to the ground state(s) of a proposed Ising spin model. When the connectivity $c > e = 2.718282$, there is no state for this system as the reweighting parameter y , which takes a similar role as the inverse temperature β in conventional statistical physics, approaches infinity; consequently the ground state energy is obtained at a finite value of y when the free energy function attains its maximum value. The minimum vertex cover size at given c is estimated using population dynamics and compared with known rigorous bounds and numerical results. The backbone size is also calculated.

PACS. 75.10.Nr Spin-glass and other random models – 89.75.-k Complex systems – 05.20.-y Classical statistical mechanics

1 Introduction

The statistical physics of spin glass systems on infinitely connected lattices (*e.g.*, the Sherrington-Kirkpatrick model) has been well understood [1] and research interest is now focused on systems of finite connectivity (FC). For the limiting example of FC random lattice, development of the cavity method has been made in recent few years [2]. In a FC random lattice, each vertex interacts only with a finite number of randomly chosen neighbours; and the local structure of the lattice is tree-like, with the shortest distance between two randomly chosen vertices diverging as the system size goes to infinity. This tree-like property makes it feasible to study the system by iterative cavity method [3]. The cavity method for n -dimensional regular lattice systems is yet to be worked out, and some aspects of such kind of systems has been understood using the well-developed mathematical tool of gauge transformation [4,5].

The zero-temperature property of random lattice spin glasses is especially interesting. Here, the cavity formalism could be greatly simplified because only the minimum energy states are relevant [6]. Furthermore, many combinatorial optimisation problems in computer sciences could be studied through a mapping into an appropriate random spin glass model, examples of which include the K-sat [7], the XOR-sat [8,9], the vertex covering problem [10,11], the number partition problem [12], and so on. Recently,

the zero-temperature cavity method was used on the random K-sat problem and the phase diagram for $k = 3$ was obtained [13,14].

In this work we hope to improve our understanding on the vertex cover problem. Consider a random graph G composed of N vertices $V = \{1, 2, \dots, N\}$. Between any two vertices an edge is present with probability $c/(N-1)$ and absent with probability $1 - c/(N-1)$, so that on average each vertex has c neighbours (*i.e.*, the average connectivity is c). The resulting edge set of graph G is denoted by $E(G)$. A vertex cover of this graph consists of a set of vertices $\Lambda = \{i_1, i_2, \dots, i_m\}$, with the property that if edge $(i, j) \in E(G)$, then either $i \in \Lambda$ or $j \in \Lambda$ or both [10]. In the general case there are many different ways to cover a graph of size N ; an interesting question is: Does there exist a vertex cover of size not exceeding xN ($0 < x < 1$)?

For large systems it was revealed that a sharp threshold value $x_c(c)$ exists. When $x > x_c(c)$, with probability approaching unity a vertex cover of size $\leq xN$ could be constructed for a given graph; while when $x < x_c(c)$ this probability approaches zero. This sharp threshold is also closely related to computational complexity. The vertex cover problem is NP-complete [15], and a time growing exponentially with N may be needed to determine whether a vertex cover of size $\leq xN$ exists or not for a given graph [16]. In the case of FC random graphs, when $x > x_c(c)$ or $x < x_c(c)$, it is relatively easy for a heuristic algorithm to check the existence of such a vertex cover; however, when $x \sim x_c(c)$, search complexity increases

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dramatically [10]. For the practical purpose of designing better algorithms, it is important for us to understand the reason of this easy-hard-easy transition and to obtain a precise estimate of the threshold value $x_c(c)$.

An rigorous bound exists for $x_c(c)$ [17]: $x_l(c) < x_c(c) < 1 - \ln c/c$, where $x_l(c)$ is the root of

$$x_l(c) \ln x_l(c) + [1 - x_l(c)] \ln[1 - x_l(c)] + (c/2)[1 - x_l(c)]^2 = 0; \quad (1)$$

furthermore, $x_c(c)$ approaches the following asymptotic form at large c [18]:

$$x_c(c) = 1 - (2/c)[\ln c - \ln \ln c + 1 - \ln 2] + o(1/c). \quad (2)$$

Using replica method of statistical physics, an analytical expression for x_c was found in [10]:

$$x_c(c) = 1 - W(c)/c - W^2(c)/2c, \quad (3)$$

where $W(c)$ is the Lambert-W-function defined by $W(c) \exp[W(c)] = c$. Equation (3) is exact when $c \leq e = 2.718282$ [19]. For $c > e$ equation (3) underestimates the true threshold value, and for $c > 20.7$ it is lower than the rigorous lower bound equation (1).

The present work focuses on the case of $c > e$. Using zero-temperature cavity method, we calculate both analytically and numerically the value of $x_c(c)$ and compare it with equations (1, 2) and with numerical calculations reported in [10]. In Section 2 an energy functional is introduced. In Section 3 the cavity formalism is outlined and the free energy expression is given. We investigate the $y \rightarrow \infty$ situation in Section 4 (y is a reweighting parameter [6, 13, 14], it plays the role of the inverse temperature β of conventional statistical mechanics). Section 5 reports the population dynamics results on the threshold value $x_c(c)$ and on the backbone size. We conclude the work in Section 6.

2 The energy functional

We attach to each vertex of the random graph G an Ising spin $\sigma = \{-1, +1\}$. Associated with each spin micro-configuration is the following energy functional

$$E[\{\sigma_i\}] = - \sum_{i=1}^N \sigma_i + \frac{\lambda}{2} \sum_{(i,j) \in E(G)} (1 + \sigma_i)(1 + \sigma_j), \quad (4)$$

where λ is a constant parameter chosen to be greater than unity [20]. Denote $N_{\text{mvc}}(G)$ as the size of the minimum vertex cover(s) of graph G , then the minimum energy over all the 2^N possible spin configurations of graph G is

$$E_{\text{min}} = 2N_{\text{mvc}}(G) - N. \quad (5)$$

Some explanation on equation (5). First, E_{min} is reachable. We denote $A_{\text{mvc}}(G)$ as (one of) the minimum-sized vertex cover(s), and assign $\sigma = -1$ to vertices in $A_{\text{mvc}}(G)$ and $\sigma = +1$ to vertices outside. The energy of this spin

configuration is $2N_{\text{mvc}}(G) - N$. Second, no spin configuration could have lower energy. To see this, suppose that another spin configuration has lower energy than equation (5). This spin configuration must contain $N_a < N_{\text{mvc}}(G)$ negative spins, with $N_a + \lambda N_0 < N_{\text{mvc}}(G)$, where $N_0 = (1/4) \sum_{(i,j) \in E(G)} (1 + \sigma_i)(1 + \sigma_j)$. However, one at most need to change the spin values of N_0 vertices from $+1$ to -1 to make the sum $\sum_{(i,j) \in E(G)} (1 + \sigma_i)(1 + \sigma_j) = 0$, and the resulting new set of negative spin vertices is a vertex cover with size $N_a + N_0 < N_{\text{mvc}}(G)$. This conflicts with our original assumption that $N_{\text{mvc}}(G)$ is the minimum vertex cover size.

The problem of finding $x_c(c)$ is converted to finding the average of E_{min}/N over the random graphs G :

$$x_c(c) = (1/2)(1 + \overline{E_{\text{min}}}/N). \quad (6)$$

Here $\overline{(\cdot)}$ means the average of (\cdot) over different realizations of the random graph. We use cavity method to estimate $\overline{E_{\text{min}}}/N$. In the next section, the cavity formalism for the present problem is outlined. The reader is referred to [1, 2, 6, 13, 14] for more detailed discussion.

3 The zero-temperature cavity formalism

At zero temperature, only the minimum energy configurations are relevant. There could be a great many energy local-minima for equation (4). For very large system size N , we group these configurations into different "states". A state of the system corresponds to a set of spin micro-configurations. These spin configurations all have the same energy, which is a local minimum of equation (4); and two such spin configurations differ only in a finite number of spin flips. The average number of states at given density ϵ of local minimum energy is assumed to be an exponentially increasing function of system size N , and is characterised by the entropy density $\Sigma(c, \epsilon)$. We can introduce a reweighting parameter y and define an zero-temperature free energy density $\Phi(y)$ through the following equation

$$\int d\epsilon \exp[-Ny\epsilon + N\Sigma(c, \epsilon)] = \exp[-Ny\Phi(y)]. \quad (7)$$

Equation (7) has the same form as the conventional definition of free energy in textbooks of equilibrium statistical physics. The reweighting parameter y plays the role of inverse temperature. A large value of y ensures that states with lower energies will be favoured, provided that such states exist (*i.e.*, $\Sigma(c, \epsilon) \geq 0$).

Suppose we have a system of N vertices (spins). Now add a new spin σ_0 into the system and connect it to k preexisting spins $\sigma_1, \dots, \sigma_k$, where k obeys the Poisson distribution of mean c , $P_P(k, c) = e^{-c} c^k / k!$. The energy of the N -spin system at fixed value of the spins $\sigma_1, \dots, \sigma_k$ is supposed to be

$$E^{(N)}(\sigma_1, \dots, \sigma_k) = A - \sum_{i=1}^k h_i \sigma_i, \quad (8)$$

with A being a constant. In the above equation, h_i is the local field (called the cavity field) felt by spin σ_i in the absence of σ_0 in a given macroscopic state. Since the graph is locally tree-like, as the system size becomes very large, the shortest distance between two randomly chosen cavity spins also becomes large; therefore, the cavity field h_i felt by spin σ_i becomes independent of the values of all the other cavity spins [21]. After the addition of spin σ_0 , the minimum energy of the $(N+1)$ -spin system at fixed value of σ_0 is

$$E(\sigma_0) = \begin{cases} A - \sigma_0, & \text{if } k = 0 \\ A - \sum_{i=1}^k \hat{w}(h_i) - [1 + \sum_{i=1}^k \hat{u}(h_i)]\sigma_0, & \text{if } k \geq 1. \end{cases} \quad (9)$$

Here,

$$\begin{aligned} \hat{w}(h) &= 0 & \text{if } h = 1 & \text{ and } = |h| & \text{if } h \leq 0, \\ \hat{u}(h) &= -1 & \text{if } h = 1 & \text{ and } = 0 & \text{if } h \leq 0. \end{aligned} \quad (10)$$

(We have used the fact that the cavity fields at zero temperature are integer-valued and do not exceed unity [21].) The energy shift caused by the addition of spin σ_0 is $\Delta E_1 = -1$ (if $k = 0$) and $\Delta E_1 = -\sum_{i=1}^k [\hat{w}(h_i) - |h_i|] - [1 + \sum_{i=1}^k \hat{u}(h_i)]$ (if $k \geq 1$).

Equation (9) indicates that the cavity field at spin σ_0 is $h_0 = 1$ (if $k = 0$) or $h_0 = 1 + \sum_{i=1}^k \hat{u}(h_i)$ (if $k \geq 1$). By definition [21], the cavity field of a spin at each state has unique value; however, its value may be different for different states. Denote the probability distribution of the cavity field at vertex i among different states as $P_i(h)$ (it is called the h-survey in Ref. [14]). Because different vertices have different local environments, the h-surveys are different for different vertices. With the introduction of the reweighting parameter y which favours lower-energy states, the h-survey at spin σ_0 is related to those of the cavity spins by

$$P_0(h) = \delta_k^0 \delta(h-1) + [1 - \delta_k^0] C \int \prod_{i=1}^k [P_i(h_i) dh_i] \times \delta \left[h - 1 - \sum_{i=1}^k \hat{u}(h_i) \right] \exp(-y\Delta E_1), \quad (11)$$

where C is a normalisation constant. Equation (11) is actually a self-consistent equation for the h-survey. A careful analysis of equation (11) leads to the following expression

$$P(h) = \begin{cases} \sum_{l=0}^{\infty} \zeta_l \delta(h+l), & \text{with probability } p_1 \\ \delta(h-1), & \text{with probability } p_2 \\ \alpha \sum_{l=0}^{\infty} \zeta_l \delta(h+l) + (1-\alpha)\delta(h-1), & \text{with probability } p_3 = 1 - p_1 - p_2 \end{cases} \quad (12)$$

where $\zeta_l \geq 0$ and $\sum_l \zeta_l = 1$; $\alpha \in (0, 1)$ is determined by certain probability distribution $\rho(\alpha)$, and

$$\begin{aligned} p_1 &= 1 - \exp(-cp_2), \\ p_2 &= \exp(-c(1-p_1)). \end{aligned} \quad (13)$$

For very large system size N , equation(7) suggests that at fixed value of y , $\bar{\Phi}(y) = \epsilon - \Sigma(c, \epsilon)/y$, with ϵ being implicitly determined by $\partial \Sigma(c, \epsilon)/\partial \epsilon = y$. An explicit expression for the free energy density could be obtained by the following way:

After the addition of spin σ_0 , the averaged number of states of the $(N+1)$ -system is

$$\exp \left[(N+1)\Sigma \left(\frac{2E(G(N)) + 2k}{N+1}, \frac{E}{N+1} \right) \right] = \int d\Delta E_1 \times P^{(1)}(\Delta E_1) \times \exp \left[N\Sigma \left(\frac{2E(G(N))}{N}, \frac{E - \Delta E_1}{N} \right) \right], \quad (14)$$

where $P^{(1)}(\Delta E_1)$ is the probability distribution function of the energy shift ΔE_1 :

$$P^{(1)}(\Delta E_1) = \delta_k^0 \delta(\Delta E_1 + 1) + (1 - \delta_k^0) \int \prod_{i=1}^k [P_i(h_i) dh_i] \delta \left[\Delta E_1 + \sum_{i=1}^k (\hat{w}(h_i) - |h_i|) + \left| 1 + \sum_{i=1}^k \hat{u}(h_i) \right| \right]. \quad (15)$$

A logarithm operation is performed on equation (14), and the resulting equation is averaged over all the possible realizations of the cavity fields and k (this average operation is denoted by an overbar in the following equation). We obtain that

$$-y\bar{\Phi}(y) = \Sigma(c, \epsilon) - y\epsilon = -c \frac{\partial \Sigma(c, \epsilon)}{\partial c} + \ln \int d\Delta E_1 P^{(1)}(\Delta E_1) \exp(-y\Delta E_1). \quad (16)$$

To compute $\partial \Sigma/\partial c$, we setup an edge between two cavity spins. The energy shift, ΔE_2 , caused by this new edge obeys the following distribution

$$P^{(2)}(\Delta E_2) = \int \prod_{i=1}^2 [dh_i P_i(h_i)] \delta \left[\Delta E_2 - \left(\min_{\sigma_1, \sigma_2} \left[\frac{\lambda}{2}(1 + \sigma_1)(1 + \sigma_2) - h_1\sigma_1 - h_2\sigma_2 \right] + |h_1| + |h_2| \right) \right]. \quad (17)$$

The averaged number of states of the new system is

$$\exp \left[N\Sigma \left(\frac{2E(G(N)) + 2}{N}, \frac{E}{N} \right) \right] = \int d\Delta E_2 P^{(2)}(\Delta E_2) \times \exp \left[N\Sigma \left(\frac{2E(G(N))}{N}, \frac{E - \Delta E_2}{N} \right) \right]. \quad (18)$$

After performing the same procedure as mentioned below equation (15), we find that

$$\frac{\partial \Sigma(c, \epsilon)}{\partial c} = (1/2) \ln \int d\Delta E_2 P^{(2)}(\Delta E_2) \exp(-y\Delta E_2). \quad (19)$$

The free energy density expression could be obtained from equations (16, 19). After taking into consideration equation (12), we arrive at the following expression for the free energy density:

$$\begin{aligned} \Phi(y) = & 2p_1 - 1 - cp_2^2 + \frac{cp_2p_3}{y} \int d\alpha \rho(\alpha) \ln(\alpha + (1-\alpha)e^{-2y}) \\ & - \frac{p_3}{y} \sum_{m=1}^{\infty} \frac{P_P(m, cp_3)}{1 - e^{-ch}} \int \prod_{i=1}^m [\rho(\alpha_i) d\alpha_i] \\ & \times \ln \left(e^{-2y} + (1 - e^{-2y}) \prod_{i=1}^m \alpha_i \right) \\ & + \frac{cp_3^2}{2y} \int \prod_{i=1}^2 [d\alpha_i \rho(\alpha_i)] \ln \left(1 - (1 - e^{-2y}) \prod_{i=1}^2 (1 - \alpha_i) \right). \end{aligned} \quad (20)$$

At given y , the energy density and the entropy density are calculated by $\epsilon = \Phi(y) - yd\Phi(y)/dy$ and $\Sigma = y^2 d\Phi(y)/dy$, respectively.

4 The $y \rightarrow \infty$ limit

At this stage, it is helpful for us to introduce an auxiliary probability distribution function called the u -survey [14]:

$$Q_i(u) = C \int dh P(h) \delta(u - \hat{u}(h)) \exp(y(\hat{w}(h) - |h|)). \quad (21)$$

Based on equation (12) we know that $Q_i(u) = \delta(u)$ (with probability p_1) or $Q_i(u) = \delta(u + 1)$ (with probability p_2), or

$$Q_i(u) = \eta \delta(u) + [1 - \eta] \delta(u + 1). \quad (22)$$

The hybrid h -survey and u -survey at a given vertex are related by $\alpha = \eta/(\eta + (1-\eta)e^y)$. The distribution of the η value in the hybrid u -survey equation (22) is governed by

$$\begin{aligned} \rho(\eta) = & \sum_{m=1}^{\infty} \frac{P_P(m, cp_3)}{1 - e^{-cp_3}} \int \prod_{i=1}^m [\rho(\eta_i) d\eta_i] \\ & \times \delta \left(\eta - 1 + \frac{e^y \prod_{l=1}^n \eta_l}{\prod_{l=1}^n [\eta_l + e^y(1 - \eta_l)] + (e^y - 1) \prod_{l=1}^n \eta_l} \right). \end{aligned} \quad (23)$$

The free energy expression of equation (20) could be rewritten in the following form

$$\begin{aligned} \Phi(y) = & p_1 - p_2 - \frac{c}{2} [(1 - p_1)^2 + (p_2)^2] \\ & + \frac{p_3(1 + cp_2)}{y} \frac{1}{\ln[\eta + e^{-y}(1 - \eta)]} \\ & - \frac{p_3[1 + c(1 - p_1)]}{y} \frac{1}{\ln[1 - \eta + e^{-y}\eta]} \\ & + \frac{cp_3^2}{2y} \frac{1}{\ln[e^{-y} + (1 - e^{-y})(\eta_1 + \eta_2 - 2\eta_1\eta_2)]}. \end{aligned} \quad (24)$$

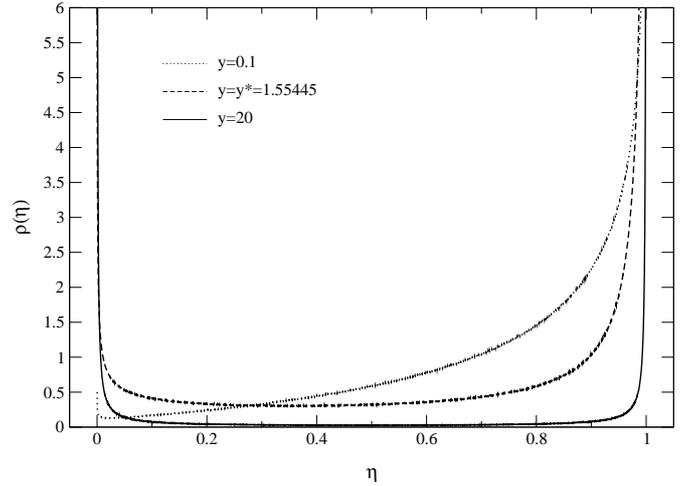


Fig. 1. The η value distribution equation (23) obtained by population dynamics at $c = 6.0$. The $\rho(\eta)$ distribution strongly depends on the value of the reweighting parameter y . In this figure, three curves at $y = 0.1$ (dotted line), $y = y^* = 1.55445$ (corresponding to the free energy maximum, dashed line), and $y = 20$ (solid line) are shown.

The overbars in equation (24) denote the average over the η distribution given by equation (23).

When $c \leq e$, the only solution of equation (13) is that $p_1 = 1 - p_2$, $p_2 = \exp(-cp_2) = W(c)/c$, and $p_3 = 0$. In this case we recover equation (3). $\Sigma(c, \epsilon) = 0$ for this solution, indicating that there is only one state. It was found that for $c \leq e$, all the vertices of the random graph could be removed by application of an leaf-removal algorithm [19].

When $c > e$ the above-mentioned solution becomes unstable, and a new solution appears with $p_2 = \exp[-c \exp(-cp_2)] < \exp(-cp_2)$, $p_1 = 1 - \exp(-cp_2)$, and $p_3 > 0$. The analysis of Bauer and Golinelli [19] reveals that a core with size proportional to N remains after application of the leaf-removal algorithm. The core is a strongly connected subgraph, in which each vertex is connected at least to two other vertices.

To estimate the value of $x_c(c)$ for $c > e$, let us first consider the limiting situation of $y \rightarrow \infty$. Equation (7) ensures that the minimum ϵ corresponds to $y = \infty$, provided that the configurational entropy is non-negative at this limit.

From equation (23) we know that as $y \rightarrow \infty$,

$$\rho(\eta) = r_1 \delta(\eta - 0^+) + r_2 \delta(\eta - 1^-) + r_3 \rho^*(\eta), \quad (25)$$

where $\rho^*(\eta)$ is a uniform distribution over $(0, 1)$. Equation (25) is confirmed by the population dynamics calculation (Fig. 1). It is easy to verify that

$$r_1 = \frac{1 - cp_2}{cp_3}, \quad r_2 = \frac{c + cp_1 - \ln c}{cp_3}, \quad r_3 = \frac{\ln c - 1}{cp_3}. \quad (26)$$

Consequently, we obtain from equation (24) that

$$\epsilon_\infty = 1 - \ln^2(c)/2c - \ln(c)/c - 3/2c,$$

$$\Sigma_\infty = -\pi^2(\ln c - 1)^2/16c. \quad (27)$$

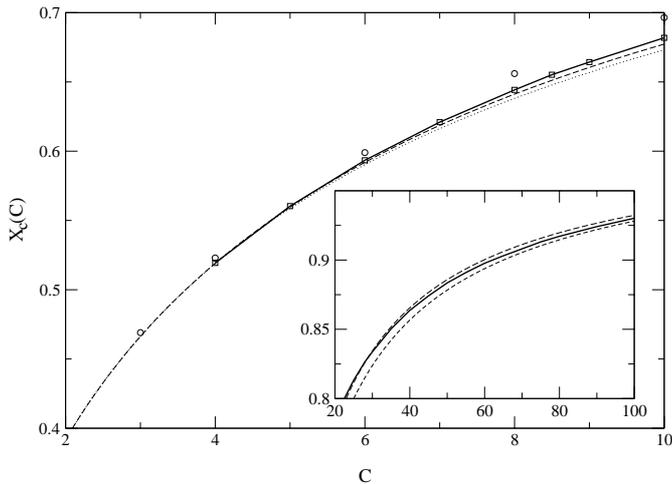


Fig. 2. The minimum vertex cover size $x_c(c)$ estimated by population dynamics (solid line and squares) and its comparison with the exact numerical enumeration values of [10] (circles), the replica-symmetric estimation equation (3) (dotted line), and the estimation given by equation (27) (dashed line). In the inset, the $x_c(c)$ value obtained by population dynamics (solid line) is compared with the rigorous lower-bound equation (1) (dashed line) and the asymptotic value equation (2) (long dashed line).

The minimum energy value given in the above equation is an improved lower-bound of the true average minimum energy. It was obtained also in [11] by replica method. However, at $y = \infty$ the entropy density Σ_∞ is negative, suggesting that there is no state at this energy density. Indeed the $x_c(c)$ value (Eq. (6)) of this solution also exceeds the rigorous lower-bound when $c > 27.3$. The true energy density must be higher than the value given in equation (27). A better estimate of the minimum energy density could be obtained by calculating the maximum value of $\Phi(y)$ with respect to the reweighting parameter y [22]. This is done in the next section with population dynamics [2].

5 Population dynamics at finite y

To obtain the value of $\Phi(y)$ at any given y , the technique of population dynamics is used [2]. A large population of η 's is generated and this population then evolves according to equation (23). The averages in equation (24) are calculated numerically. The resulting estimates of $x_c(c)$ are shown in Figure 2 and listed in Table 1.

The threshold curve $x_c(c)$ obtained by the present method lies within the rigorous bound given by equation (1). It is therefore an improved estimate compared with equation (3) and equation (27), both of which exceed the rigorous lower-bound when c is larger than certain value. However, the values of x_c estimated by the cavity method is systematically smaller than enumeration results on finite systems (Fig. 2). When the average connectivity c is large, the estimated $x_c(c)$ value approaches the asymptotic value but it lies below the asymptotic curve (Fig. 2 inset). These discrepancies suggest that the $x_c(c)$ threshold value obtained by the cavity method is not exact; it

Table 1. The $x_c(c)$ value obtained by the population dynamics (column 2) and its comparison with the exact numerical enumeration value reported in [10] (column 4). The reweighting parameter value y^* of column 3 at given c corresponds to the maximum of the free energy density equation (24). To determine y^* , equation (24) is fitted with two adjustable parameters: $\Phi(y) = \epsilon_\infty + C_1/y + C_2 \exp(-y)/y$, where ϵ_∞ is given in equation (27). The population size adopted in the present work is 20,000.

c	x_c (cavity)	y^*	x_c (enumeration)
4.0	0.5194	1.3093	0.523 ± 0.001
5.0	0.5603	1.5758	
6.0	0.5934	1.5545	0.599 ± 0.001
7.0	0.6210	1.5260	
8.0	0.6443	1.5259	0.656 ± 0.003
9.0	0.6643	1.5326	
10.0	0.6819	1.5434	0.697 ± 0.003

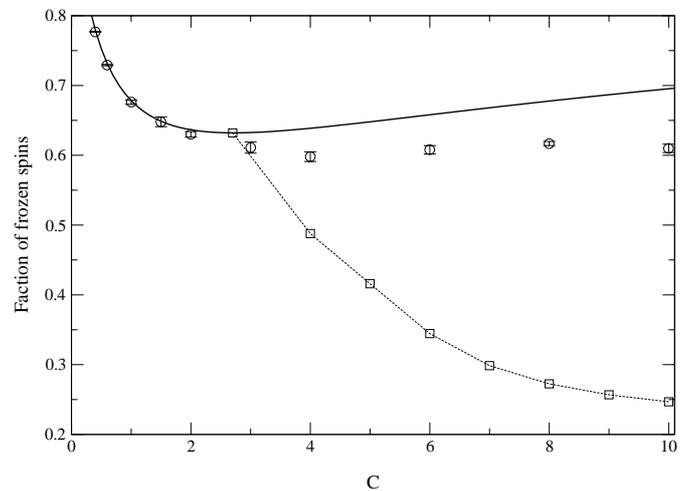


Fig. 3. Fraction of frozen spins calculated by population dynamics [23] (squares) and its comparison with numerical result reported in [11] and the replica-symmetric result [11] (thick solid line).

could serve as an improved lower-bound of the real threshold curve. The reason for the failure of the cavity method to obtain exact threshold values for $c > e$ are discussed in the next section.

The cavity field distribution equation (12) contains information on the spin state at a given vertex. A spin σ_i will be fixed at $\sigma_i = +1$ if the local cavity field is distributed according to $P(h_i) = \delta(h - 1)$ or $P(h_i) = \alpha \sum_{l=0}^{\infty} \zeta_l \delta(h + l) + (1 - \alpha) \delta(h - 1)$ but with $\alpha \rightarrow 0^+$ [23]; on the other hand, σ_i will be fixed at $\sigma_i = -1$ with probability $\sum_{l=1}^{\infty} \zeta_l$ if $P(h_i) = \sum_{l=0}^{\infty} \zeta_l \delta(h + l)$ or $P(h_i) = \alpha \sum_{l=0}^{\infty} \zeta_l \delta(h + l) + (1 - \alpha) \delta(h - 1)$ but with $\alpha \rightarrow 1^-$ [23]. For the minimum vertex covers at $y = y^*$, the probability for a randomly chosen vertex to have fixed spin value is calculated by population dynamics (see Fig. 3). The fraction of frozen spins calculated by the population dynamics is much lower than that obtained by numerical enumeration [11]. This discrepancy may be further indication that the full hierarchy of replica symmetry breaking is needed

to completely describe the property of the minimum vertex covers. We have noticed that the fraction of frozen spins strongly depends on the value of the reweighting parameter y . When $y \rightarrow \infty$ the replica-symmetric result of [11] is recovered; while for $y \sim 2y^* - 3y^*$ the data obtained by numerical enumeration is approached.

6 Discussion

In this work, we estimate the average minimum vertex cover size $x_c(c)$ for random graphs of finite connectivity $c > e$ in the large N limit. The obtained $x_c(c)$ curve lies within the rigorous bound [17] and approaches the asymptotic curve equation (2) at large c values. It could be regarded as an improved lower-bound of the real threshold $x_c(c)$ value.

When $c > e$ the threshold x_c estimated by the cavity method is not exact. The reason may be the following: The cavity method is equivalent to one-step replica-symmetry-breaking [1, 2, 6], and the cavity fields on different vertices are considered as uncorrelated. Because of the core percolation beyond $c = e$ [19] in the random graph, the cavity fields of different vertices may actually be correlated strongly. To partly account for this effect, one possibility is to consider also non-integer cavity fields. We hope to return to this point in a latter work.

The cavity method has inspired very efficient algorithms to tackle the random K-sat problem. In the random K-sat problem, there exists a glassy phase at $y \rightarrow \infty$ for certain range of the connectivity c [13, 14]. In this glassy phase, the minimum energy is still located at $y = \infty$ but the complexity is positive, and the algorithm based on the idea of cavity field [14] works well in this phase. For the vertex cover problem, the present work suggests that such a $y = \infty$ glass phase does not exist (this conclusion seems also true for the vertex cover problem on random hypergraphs where each “edge” is a triangle [24]). This indicates that vertex covers of size $Nx_c(c) + O(1)$ are extremely few. It remains to be seen whether or not algorithms based on cavity method could efficiently find vertex covers of size slightly beyond $Nx_c(c)$ for a given random graph.

This work is made possible by a post-doctoral fellowship of the Max Planck Society. I thank M. Weigt for suggestions concerned with earlier versions of this manuscript, and A.K. Hartmann and M. Weigt for sharing numerical data. I also thank professor R. Lipowsky and professor Yu Lu for support.

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20. The value of λ is irrelevant as long as $\lambda > 1$: The bond energy, $(\lambda/2)(1 + \sigma_i)(1 + \sigma_j)$, equals 0 or 2λ ; if it is 2λ , one can choose either $\sigma_i = -1$ or $\sigma_j = -1$ and the total energy will be lowered. However, choosing $\lambda \simeq 1$ might facilitate a search algorithm in finding the true energy minimum
21. Some more discussion about the cavity fields. At zero temperature, each state of the system corresponds to an ensemble of spin microscopic configurations. All these configurations have the same energy (corresponding to a minimum value of the energy functional Eq. (4)), and only finite number of spin flips is needed to transit from one to another configuration of them. The spins on some of the vertices are fixed at $\sigma = +1$ or $\sigma = -1$ in all these configurations, while the spins on the remaining vertices are not fixed. We say that those spins whose value fluctuates from one configuration to another do not feel any magnetic field in this state ($h = 0$). A spin whose value is fixed feels a non-zero magnetic field. The magnitude of this non-zero cavity field is defined as half the total increase in minimum energy associated with flipping this spin. For our present problem, an energy local minimum has value $2N_{\downarrow} - N$, where N_{\downarrow} is the total number of negatively valued spins. Therefore, we conclude that (1) all the cavity fields must be integer-valued, and (2) the cavity fields could not exceed unity ($h \leq 1$)
22. This is understood by the following consideration: According to equation (7), to obtain the energy absolute minimum the reweighting parameter y should be set to its largest possible value. On the other hand, $\Sigma = y^2 d\Phi(y)/dy$ changes sign from positive to negative as y is increased beyond some value. Therefore the largest y value is obtained by the equation $d\Phi(y)/dy = 0$, which corresponds to the maximum of $\Phi(y)$
23. In the numerical calculation, $\alpha \rightarrow 0^+$ is replaced by the condition $\alpha \leq \alpha_c$ and $\alpha \rightarrow 1^-$ is replaced by $\alpha \geq 1 - \alpha_c$ with α_c chosen to be 0.01. It turns out that the calculated fraction of frozen spins is sensitive to the α_c value. A more stringent criterion (say $\alpha_c = 0.001$) leads to even smaller fraction, while a looser criterion (say $\alpha_c = 0.05$) leads to much bigger value
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